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Geometry of the Submanifolds of ESX_n. III. Parallelism in ESX_n and Its Submanifolds

Kyung Tae Chung[1] **and Keum Sook So**[2]

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A connection which is both Einstein and semisymmetric is called an *ES* connection. And a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through an *ES* connection, is called an n-dimensional *ES* manifold and denoted by ESX_n. This paper is the third part of a systematic study of the submanifolds X_m of ESX_n. In the first part, we introduced a new concept of the *C*-nonholonomic frame of reference in ESX_n at points of X_m and dealt with its consequences. In the second part, the generalized fundamental equations on a hypersubmanifold of ESX_n were derived as an application of the *C*-nonholonomic frame of reference. The purpose of the present paper is to study parallelism in ESX_n and in its submanifold X_m, using the *C*-nonholonomic frame of reference and the new concept of ES_i curves.

1. INTRODUCTION

Einstein (1950, Appendix II) proposed a unified field theory that, while physically motivated, is mainly geometrical, in that it mainly consists of a set of geometrical postulates for the space-time X_4. He did not extensively develop the geometrical consequences of these postulates. Characterizing Einstein's four-dimensional unified field theory as a set of geometrical postulates for X_4, Hlavatý (1957) provided its mathematical foundation. Since then the geometrical consequences of these postulates have been further developed. A number of mathematicians and theoretical physicists have contributed to the development of this theory. The generalization of this theory to the n-dimensional generalized Riemannian manifold X_n has also been attempted by, e.g., Wrede (1958), Mishra (1962), and Chung *et al.* (1981*a*,*b*, 1985). However, the main problem for the n-dimensional case is that it is unable to display a surveyable tensorial solution of the Einstein

[1]Department of Mathematics, Yonsei University, Seoul, Korea.
[2]Department of Mathematics, Hanlim University, Choonchun, Korea.

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equation in terms of $g_{\lambda\mu}$, probably due to the complexity of the higher dimensions.

Recently, Chung *et al.* (1987) introduced the concept of an n-dimensional ES manifold ESX_n, imposing the semisymmetric condition on X_n, and found a unique representation of the Einstein connection in a beautiful and surveyable form. Later, Chung *et al.* (1988*a*,*b*) also investigated the curvature theory and field equations in ESX_n.

This paper is the third part of a systematic study of the submanifolds X_m of ESX_n. In the first part (Chung *et al.*, 1989*a*), a new concept of the C-nonholonomic frame of reference in ESX_n at points of X_m was introduced and its consequences were considered. In the second part, Chung *et al.* (1989*b*) derived the generalized fundamental equations on an ES hypersubmanifold as an application of the C-nonholonomic frame of reference. The purpose of the present paper is to study parallelism in ESX_n and its submanifold X_m. This paper contains five sections. Section 2 introduces some preliminary notations, concepts, and results. Section 3 deals with parallelism in a general X_n. Most results in this section are well known. Section 4 is devoted exclusively to parallelism in ESX_n, investigating properties of the tensor $U^{\nu}{}_{\lambda\mu}$, the vectors S_λ and U_λ, and the ES_i curves. In the last section we discuss parallelism on the submanifold X_m of X_n, and then of ESX_n, using the C-nonholonomic frame of reference and the new concept of ES_i curves.

All considerations in the present paper are for a general $n > 1$ and for all possible classes and indices of inertia.

2. PRELIMINARIES

This section is a brief collection of basic concepts, results, and notations needed in the present paper. It is based on the results and notation of Hlavatý (1957) for Section 2.1, Chung *et al.* (1987) for Section 2.2 and Chung *et al.* (1989*a*) for Section 2.3.

2.1. *n*-Dimensional Unified Field Theory on X_n

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system y^ν, with coordinate transformation $y^\nu \to \bar{y}^\nu$,[3] for which

$$\mathrm{Det}\left(\frac{\partial \bar{y}}{\partial y}\right) \neq 0 \tag{2.1}$$

The n-dimensional unified field theory on X_n is an n-dimensional generalization of Einstein's unified field theory on the space-time X_4. Therefore,

[3]Throughout the present paper, lowercase Greek indices are used for the holonomic components of tensors in X_n. They take the values $1', 2', \ldots, n'$ and follow the summation convention.

in this theory *the algebraic structure on* X_n is imposed by a general real nonsymmetric tensor $g_{\lambda\mu}$, the so-called *Einstein unified field tensor*. It may be decomposed into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\mathfrak{g} = \mathrm{Det}(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = \mathrm{Det}(h_{\lambda\mu}) \neq 0 \tag{2.3}$$

Hence, we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu \tag{2.4}$$

The tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ will serve for raising and/or lowering indices of holonomic components of tensors in X_n in our further considerations.

The manifold X_n is assumed to be connected by a real general connection $\Gamma_{\lambda\mu}^{\nu}$ with the following transformation rule:

$$\bar{\Gamma}_{\lambda\mu}^{\nu} = \frac{\partial \bar{y}^\nu}{\partial y^\alpha}\left(\frac{\partial y^\beta}{\partial \bar{y}^\lambda}\frac{\partial y^\gamma}{\partial \bar{y}^\mu}\Gamma_{\beta\gamma}^{\alpha} + \frac{\partial^2 y^\alpha}{\partial \bar{y}^\lambda\, \partial \bar{y}^\mu}\right) \tag{2.5}$$

Furthermore, in the n-dimensional unified field theory, *the differential geometric structure* is imposed on X_n by the tensor $g_{\lambda\mu}$ by means of the *Einstein connection* $\Gamma_{\lambda\mu}^{\nu}$ satisfying a system of Einstein's equations

$$\partial_\omega g_{\lambda\mu} - \Gamma_{\lambda\omega}^{\alpha} g_{\alpha\mu} - \Gamma_{\omega\mu}^{\alpha} g_{\lambda\alpha} = 0 \tag{2.6a}$$

This system is shown to be equivalent to the tensorial form

$$D_\omega g_{\lambda\mu} = 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.6b}$$

where $S_{\lambda\mu}{}^{\nu} = \Gamma_{[\lambda\mu]}^{\nu}$ is the torsion tensor of $\Gamma_{\lambda\nu}^{\nu}$ and D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_{\lambda\mu}^{\nu}$.

A procedure similar to Christoffel elimination applied to the symmetric part of (2.6b) yields that if the system (2.6) admits a solution $\Gamma_{\lambda\mu}^{\nu}$, it must be of the form

$$\Gamma_{\lambda\mu}^{\nu} = \{{}_\lambda{}^\nu{}_\mu\} + S_{\lambda\mu}{}^{\nu} + U^\nu{}_{\lambda\mu} \tag{2.7}$$

where

$$U^\nu{}_{\lambda\mu} = 2 h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} \tag{2.8}$$

and $\{{}_\lambda{}^\nu{}_\mu\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.

An eigenvector V^ν of $k_{\lambda\mu}$ which satisfies

$$(M h_{\lambda\mu} + k_{\lambda\mu}) V^\mu = 0 \qquad (M \text{ is a scalar}) \tag{2.9}$$

is called a basic vector of X_n, and the corresponding eigenvalue M of $k_{\lambda\mu}$ a basic scalar of X_n.

2.2. The *ES* Manifold ESX_n

A connection $\Gamma_{\lambda\mu}^{\nu}$ is said to be *semisymmetric* if its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form

$$S_{\lambda\mu}{}^{\nu}=2\delta_{[\lambda}{}^{\nu}X_{\mu]} \tag{2.10}$$

for an arbitrary vector X_μ. A semisymmetric connection which satisfies (2.6) is called an *ES* connection. A generalized Riemannian manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through an *ES* connection is called an *n-dimensional ES manifold* and will be denoted by ESX_n.

It has been shown that there always exists a *unique* n-dimensional *ES* connection $\Gamma_{\lambda\mu}^{\nu}$ of the form

$$\Gamma_{\lambda\mu}^{\nu}=\{{}_{\lambda}{}^{\nu}{}_{\mu}\}+2k_{(\lambda}{}^{\nu}X_{\mu)}+2\delta_{[\lambda}{}^{\nu}X_{\mu]} \tag{2.11}$$

for a unique vector X_μ represented by [4]

$$X_\lambda=\frac{1}{n-1}\,{}^*h^{\alpha\beta}\nabla_\alpha k_{\beta\lambda} \tag{2.12}$$

where ∇_λ is the symbolic vector of the covariant derivative with respect to $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$. Therefore, we note that *there always exists one and only one* ESX_n once a unified field tensor $g_{\lambda\mu}$ is given.

2.3. The *C*-Nonholonomic Frame of Reference

Let X_m be a submanifold of a generalized n-dimensional Riemannian manifold X_n $(m<n)$, defined by a system of real parametric equations

$$y^\nu=y^\nu(x^1,\ldots,x^m) \tag{2.13}$$

It is assumed that the functions $y^\nu(x^i)$ are sufficiently differentiable and the rank of the matrix of derivatives $B_i^\nu=\partial y^\nu/\partial x^i$ is m. Clearly X_m is an m-dimensional differentiable manifold in its own right. Furthermore, if a vector field tangential to X_m is given by U^ν in the y's and U^i in the x's, respectively, we must have

$$U^\nu=B_i^\nu U^i \tag{2.14}$$

Since the rank of the matrix (B_i^ν) is m, the condition (2.3) guarantees the existence of *the first set* of $n-m$ nonnull real vectors $\underset{x}{N}{}^\nu$ normal to X_m,

[4]The precise representation of the tensor ${}^*h^{\lambda\nu}$ is given in Chung *et al.* (1981*a*).

which are linearly independent and mutually orthogonal. That is,

$$h_{\alpha\beta}B_i^\alpha \underset{x}{N}^\beta=0, \qquad h_{\alpha\beta}\underset{x}{N}^\alpha \underset{x}{N}^\beta=0 \qquad \text{for} \quad x\neq y \tag{2.15a}$$

The process of determining this set is not unique unless $m=n-1$. However, we may choose their magnitudes such that

$$h_{\alpha\beta}\underset{x}{N}^\alpha \underset{x}{N}^\beta=\varepsilon_x \tag{2.15b}$$

where $\varepsilon_x=\pm 1$ accordingly as the left-hand side of (2.15b) is positive or negative.

Put

$$E_A^\nu=\begin{cases} B_i^\mu & \text{if} \quad A=1,\ldots,m \quad (=i) \\ \underset{x}{N}^\nu & \text{if} \quad A=m+1,\ldots,n \quad (=x) \end{cases} \tag{2.16}$$

Since $\{E_A^\nu\}$ is a set of n linearly independent vectors in X_n at points of X_m, there exists a unique *second* set $\{E_\lambda^A\}$ of n linearly independent vectors at points of X_m such that[5]

$$E_\lambda^A E_A^\nu=\delta_\lambda^\nu, \qquad E_\alpha^A E_B^\alpha=\delta_B^A \tag{2.17}$$

Put

$$E_\lambda^A=\begin{cases} B_\lambda^i & \text{if} \quad A=1,\ldots,m \quad (=i) \\ \overset{x}{N}_\lambda & \text{if} \quad A=m+1,\ldots,n \quad (=x) \end{cases} \tag{2.18}$$

$$B_\lambda^\nu=B_\lambda^i B_i^\nu \tag{2.19}$$

Then, it has been shown that the following relations hold in virtue of (2.17):

$$B_\alpha^i B_j^\alpha=\delta_j^i, \qquad \overset{x}{N}_\alpha \underset{y}{N}^\alpha=\delta_y^x, \qquad B_\alpha^i \underset{x}{N}^\alpha=\overset{x}{N}_\alpha B_i^\alpha=0 \tag{2.20}$$

$$B_\lambda^\nu=\delta_\lambda^\nu-\sum_x \overset{x}{N}_\lambda \underset{x}{N}^\nu, \qquad B_\lambda^\alpha \overset{x}{N}_\alpha=B_\alpha^\nu \underset{x}{N}^\alpha=0 \tag{2.21}$$

In virtue of (2.21), we note that the vectors B_λ^i form *the second set* of linearly independent vectors tangential to X_m. We also note that the set

[5]In the present paper, we use the following different types of indices: (a) Lowercase Greek indices $\alpha, \beta, \gamma, \ldots$, running from $1'$ to n' are used for the holonomic components of tensors of X_n. (b) Capital Latin indices $A, B, C, \ldots$, running from 1 to n are used for the C-nonholonomic components of tensors in X_n at points of X_m. (c) Lowercase Latin indices $i, j, k, \ldots$, with the exception of x, y, z, running from 1 to m ($<n$). (d) Lowercase indices x, y, z, running from $m+1$ to n. The summation convention is operative with respect to each set of the above indices within their range, with the exception of x, y, z.

$\{\overset{x}{N}_\lambda\}$ is *the second set* of $n-m$ nonnull vectors normal to X_m, which are linearly independent and mutually orthogonal.

The sets E_A^ν and E_λ^A will be referred to as a *C-nonholonomic frame of reference in* X_n at points of X_m. This frame of reference gives rise to *C-nonholonomic components* of a tensor in X_n: If $T_{\lambda\ldots}^{\nu\ldots}$ are holonomic components of a tensor in X_n, then at points of X_m its C-nonholonomic components $T_{B\ldots}^{A\ldots}$ are defined by

$$T_{B\ldots}^{A\ldots} = T_{\beta\ldots}^{\alpha\ldots} E_\alpha^A \cdots E_B^\beta \cdots \tag{2.22}$$

In virtue of (2.17), an easy inspection shows that

$$T_{\lambda\ldots}^{\nu\ldots} = T_{B\ldots}^{A\ldots} E_A^\nu \cdots E_\lambda^B \cdots \tag{2.23}$$

As a consequence of (2.23), we have

$$h_{\lambda\mu} = h_{ij} B_\lambda^i B_\mu^j + \sum_x \varepsilon_x \overset{x}{N}_\lambda \overset{x}{N}_\mu \tag{2.24a}$$

$$h^{\lambda\nu} = h^{ij} B_i^\lambda B_j^\nu + \sum_x \varepsilon_x \underset{x}{N}^\lambda \underset{x}{N}^\nu \tag{2.24b}$$

As another consequence of (2.23), we have

$$T^\nu = T^i B_i^\nu + \sum_x T^x \underset{x}{N}^\nu \tag{2.25a}$$

or equivalently

$$T_\lambda = T_i B_\lambda^i + \sum_x T_x \overset{x}{N}_\lambda \tag{2.25b}$$

where T^ν are components in the y's of a vector in X_n. Equations (2.25a) and (2.25b) show that at each point of X_m any vector T^ν in X_n may be expressed as the sum of two vectors, the former tangential to X_m, the latter normal to X_m. Furthermore, T^i, called *the induced vector on* X_m *of* T^ν *in* X_n, are components of a tangent vector to X_m at points of X_m relative to the coordinate transformations $x^i \to \bar{x}^i$. This concept may be generalized to an arbitrary tensor $T_{\lambda\ldots}^{\nu\ldots}$ in X_n in like manner. If $T_{\lambda\ldots}^{\nu\ldots}$ are the components in the y's of a tensor in X_n, then the quantities

$$T_{j\ldots}^{i\ldots} = T_{\beta\ldots}^{\alpha\ldots} B_\alpha^i \cdots B_j^\beta \cdots \tag{2.26}$$

evaluated at points of X_m are called *the components of the induced tensor on* X_m of $T_{\lambda\ldots}^{\nu\ldots}$ in X_n. In fact, they are components of a tensor in X_m relative to the coordinate transformations $x^i \to \bar{x}^i$.

It has been shown that the induced tensors h_{ij} of $h_{\lambda\mu}$ and h^{ik} of $h^{\lambda\nu}$ satisfy

$$h_{ij}h^{ik}=\delta_j^k \tag{2.27}$$

Therefore, they may be used for raising and/or lowering indices of the induced tensors on X_m in the usual manner.

2.4. The Induced Connection on X_m of X_n

If $\Gamma^{\nu}_{\lambda\mu}$ is a connection on X_n, the connection Γ^k_{ij} defined by

$$\Gamma^k_{ij}=B^k_\gamma(B^\gamma_{ij}+\Gamma^\gamma_{\alpha\beta}B^\alpha_i B^\beta_j), \qquad B^\gamma_{ij}=\frac{\partial B^\gamma_i}{\partial x^j}=\frac{\partial^2 y^\gamma}{\partial x^i\,\partial x^j} \tag{2.28}$$

is called the *induced connection* of $\Gamma^{\nu}_{\lambda\mu}$ on X_m of X_n.

Each of the following statements has been already proved:

(a) The torsion tensor $S_{ij}{}^k$ of the induced connection Γ^k_{ij} is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma^\nu_{\lambda\mu}$. That is,

$$S_{ij}{}^k=S_{\alpha\beta}{}^\gamma B^\alpha_i B^\beta_j B^k_\gamma \tag{2.29}$$

(b) The induced connection $\left\{ {k \atop i\ j} \right\}$ of $\left\{ {\nu \atop \lambda\ \mu} \right\}$ is the Christoffel symbol defined by h_{ij}. That is,

$$\left\{ {k \atop i\ j} \right\}=\tfrac{1}{2}h^{kp}(\partial_i h_{jp}+\partial_j h_{ip}-\partial_p h_{ij})$$

(c) On an X_m of ESX_n, the induced connection Γ^k_{ij} is of the form

$$\Gamma^k_{ij}=\left\{ {k \atop i\ j} \right\}+2\delta_{[i}{}^k X_{j]}+2k_{(i}{}^k X_{j)} \tag{2.31}$$

Hence, the induced connection is also semisymmetric.

(d) A necessary and sufficient condition for the induced connection Γ^k_{ij} to be Einstein is[6]

$$\sum_x k_{x[i}\overset{x}{\Omega}_{j]k}=0 \tag{2.32}$$

3. PARALLELISM IN X_n

In this section we investigate parallelism and paths in X_n. Some of the results introduced in the present section are well known.

In a general X_n there is no basis for the comparison of vectors at different points. For a Riemannian manifold, parallelism of vectors, as defined by Levi-Civita, serves as a basis for such comparison. This definition may be

[6]This condition has been proved in Theorem 3.17 of Chung and Kim (1991).

generalized for a connected generalized Riemannian manifold X_n. Let C be *any* curve in X_n, given by

$$y^\nu = y^\nu(t) \tag{3.1}$$

Definition 3.1. A vector field V^ν is said to be *parallel along C with respect to a connection* $\Gamma^{\ \nu}_{\lambda\mu}$ if it satisfies the following condition:

$$\frac{dy^\alpha}{dt} V^{[\lambda} D_\alpha V^{\nu]} = 0, \qquad V^\nu \neq \rho \frac{dy^\alpha}{dt} D_\alpha V^\nu \tag{3.2a}$$

for some $\rho \neq 0$, or equivalently

$$V^{[\lambda}\left(\frac{dV^{\nu]}}{dt} + \Gamma^{\ \nu]}_{\beta\alpha} V^\beta \frac{dy^\alpha}{dt}\right) = 0, \qquad V^\nu \neq \rho \frac{dy^\alpha}{dt} D_\alpha V^\nu \tag{3.2b}$$

In particular the curves whose tangents are parallel along themselves are called the *paths* in X_n with respect to $\Gamma^{\ \nu}_{\lambda\mu}$. A path with respect to $\{{}_\lambda{}^\nu{}_\mu\}$ is called a *geodesic* of X_n.

Therefore, the equations of paths are given by

$$\frac{dy^{[\lambda}}{dt}\left(\frac{d^2 y^{\nu]}}{dt^2} + \Lambda^{\ \nu]}_{\alpha\beta} \frac{dy^\alpha}{dt}\frac{dy^\beta}{dt}\right) = 0, \qquad \Lambda^{\ \nu}_{\alpha\beta} = \Gamma^{\ \nu}_{(\alpha\beta)} \tag{3.3}$$

Theorem 3.2. A necessary and sufficient condition that parallelism be the same along every curve in X_n with respect to two connections $\Gamma^{\ \nu}_{\lambda\mu}$ and $\bar{\Gamma}^{\ \nu}_{\lambda\mu}$ is that they are related by

$$\bar{\Gamma}^{\ \nu}_{\lambda\mu} = \Gamma^{\ \nu}_{\lambda\mu} + 2\delta^\nu_\lambda p_\mu \tag{3.4}$$

or equivalently

$$\bar{\Lambda}^{\ \nu}_{\lambda\mu} = \Lambda^{\ \nu}_{\lambda\mu} + 2\delta_{(\lambda}{}^\nu p_{\mu)} \tag{3.5a}$$

$$\bar{S}_{\lambda\mu}{}^\nu = S_{\lambda\mu}{}^\nu + 2\delta_{[\lambda}{}^\nu p_{\mu]} \tag{3.5b}$$

where p_λ is an arbitrary vector.

Proof. If parallelism is the same along every curve in X_n with respect to $\Gamma^{\ \nu}_{\lambda\mu}$ and $\bar{\Gamma}^{\ \nu}_{\lambda\mu}$, we must have (3.2b) and the corresponding one for $\bar{\Gamma}^{\ \nu}_{\lambda\mu}$. Subtraction of the former from the latter gives

$$V^{[\lambda} A_{\beta\alpha}{}^{\nu]} V^\beta \frac{dy^\alpha}{dt} = \delta_\gamma{}^{[\lambda} A_{\beta\alpha}{}^{\nu]} V^\gamma V^\beta \frac{dy^\alpha}{dt} = 0 \tag{3.6}$$

where

$$A_{\beta\alpha}{}^{\nu} = \bar{\Gamma}^{\nu}_{\beta\alpha} - \Gamma^{\nu}_{\beta\alpha} \tag{3.7}$$

is a tensor. In order that (3.6) holds for every curve and vector V^{ν} in X_n, we must have

$$\delta_{(\gamma}{}^{[\gamma}A_{\beta)\alpha}{}^{\nu]} = 0 \tag{3.8}$$

Contracting for λ and γ in (3.8), it follows that

$$A_{\beta\alpha}{}^{\nu} = 2\delta_{\beta}{}^{\nu}p^{\alpha} \tag{3.9}$$

where p_{α} is a vector defined by

$$p_{\alpha} = \frac{1}{2n} A_{\gamma\alpha}{}^{\gamma} \tag{3.10}$$

The relation (3.4) immediately follows from (3.7) and (3.9). On the other hand, it may be easily seen that the corresponding condition to (3.2b) for $\Gamma^{\nu}_{\lambda\mu}$ follows from (3.4) and (3.2b). This proves the converse statement of our assertion. The relations (3.5) are an immediate decomposition of (3.4).

Remark 3.3. In virtue of (3.4), we note that parallelism along every curve in X_n cannot be the same for two different symmetric connections. Furthermore, equation (3.3) suggests that paths in X_n are the same with respect to two connections, one of which is $\Gamma^{\nu}_{\lambda\mu}$ and the other $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is defined by (3.5a) and an arbitrary choice of $\bar{S}_{\lambda\mu}{}^{\nu}$.

The following theorem gives a condition of parallelism in terms of the covariant components of vectors.

Theorem 3.4. The condition (3.2a) of parallelism is equivalent to

$$\frac{dy^{\alpha}}{dt}(D^{\alpha}V_{[\omega)})V_{\mu]} = \frac{dy^{\alpha}}{dt} T_{\omega\mu\alpha} \tag{3.11}$$

where

$$T_{[\omega\mu]\alpha} = V^{\beta}(D_{\alpha}h_{\beta[\omega)})V_{\mu]} \tag{3.12}$$

Proof. Multiplying by $h_{\lambda\mu}h_{\nu\omega}$ on both sides of (3.2a), we have

$$\frac{dx^{\alpha}}{dt} V_{[\mu}h_{\omega]\beta}D_{\alpha}V^{\beta} = 0 \tag{3.13}$$

Our assertion (3.11) immediately follows by substituting the obvious relation

$$h_{\omega\beta}D_{\alpha}V^{\beta} = D_{\alpha}V_{\omega} - V^{\beta}D_{\alpha}h_{\beta\omega}$$

into (3.13) and making use of (3.12).

4. PARALLELISM IN ESX_n

Agreement 4.1. In this section and in what follows, we suppose that the unique vector X_λ given by (2.12) satisfies the following conditions:

(a) $X_\lambda \neq 0$.

(b) X_λ is not a gradiant vector.

Section 4.1 is concerned with some useful relations satisfied by the tensor $U^\nu{}_{\lambda\alpha}$ and the vectors

$$S_\lambda = S_{\lambda\alpha}{}^\alpha, \qquad U_\lambda = U^\alpha{}_{\lambda\alpha} \tag{4.1}$$

which are needed in our further considerations. Section 4.2 is devoted to the parallelism, paths, and parallelism-preserving change of connections in ESX_n.

4.1. The Tensor $U^\nu{}_{\lambda\mu}$ and the Vectors S_λ, U_λ

The tensor $U^\nu{}_{\lambda\mu}$ is closely related with the theory of parallelism.

Theorem 4.2. In ESX_n the tensor $U^\nu{}_{\lambda\mu}$ satisfies the following conditions:

(a) $U^\nu{}_{\lambda\mu} \neq 0$.

(b) $U_{(\nu\lambda\mu)} = 0$.

Proof. The condition (b) is a direct consequence of

$$U^\nu{}_{\lambda\mu} = 2k_{(\lambda}{}^\nu X_{\mu)} \tag{4.2}$$

which follows from (2.8) and (2.10). In order to prove statement (a), assume that $U^\nu{}_{\lambda\mu} = 0$. Then (4.2) gives

$$k_{\lambda\nu} X_\mu + k_{\mu\nu} X_\lambda = 0 \qquad \text{for every } \lambda,\ \mu,\ \nu \tag{4.3}$$

In virtue of the first condition of Agreement 4.1, there exists at least one *fixed index* ξ such that $X_\xi \neq 0$. Hence, we have

$$k_{\lambda\nu} X_\xi + k_{\xi\nu} X_\lambda = 0 \qquad \text{for every } \lambda \text{ and } \nu \tag{4.4}$$

Putting $\lambda = \xi$ in (4.4), we first note that $k_{\xi\nu} = 0$ for every ν. Hence, substituting $k_{\xi\nu} = 0$ into (4.4), we finally have

$$k_{\lambda\nu} = 0 \qquad \text{for every } \lambda \text{ and } \nu$$

which is a contradiction to the nonsymmetry of $g_{\lambda\mu}$.

Remark 4.3. Substituting (2.8) into the left-hand side of condition (b) of Theorem 4.2, we note that (b) also holds in X_n.

Theorem 4.4. In ESX_n the vectors S_λ and U_λ are given by

$$S_\lambda=(1-n)X_\lambda \tag{4.5}$$

$$U_\lambda=\tfrac{1}{2}\partial_\lambda \ln g=k_\lambda{}^\alpha X_\alpha, \qquad g=\mathfrak{g}/\mathfrak{h} \tag{4.6}$$

Proof. Putting $\mu=\nu$ in (2.10), we have (4.5). Similarly, the second relation of (4.6) may be obtained by putting $\mu=\nu$ in (4.2). According to (2.3), we note that there exists a unique tensor

$$*g^{\lambda\nu}=\frac{\partial \ln \mathfrak{g}}{\partial g_{\lambda\nu}} \tag{4.7a}$$

satisfying the condition

$$g_{\lambda\mu}\,{*g}^{\lambda\nu}=g_{\mu\lambda}\,{*g}^{\nu\lambda}=\delta_\mu^\nu \tag{4.7b}$$

In order to prove the first relation of (4.6), multiply by $*g^{\lambda\mu}$ on both sides of (2.6a) and make use of (4.7b) to derive

$$\partial_\omega \ln \mathfrak{g}-\Gamma^{\alpha}_{\alpha\omega}-\Gamma^{\alpha}_{\omega\alpha}=0 \tag{4.8a}$$

or

$$\partial_\omega \ln \mathfrak{g}+2S_\omega-2\Gamma^{\alpha}_{\omega\alpha}=0 \tag{4.8b}$$

On the other hand, in virtue of the well-known classical relation

$$\left\{{}_{\alpha}{}^{\alpha}{}_{\mu}\right\}=\tfrac{1}{2}\partial_\mu \ln \mathfrak{h}$$

we have

$$\Gamma^{\alpha}_{\omega\alpha}=\tfrac{1}{2}\partial_\omega \ln \mathfrak{h}+\mathrm{S}_\omega+\mathrm{U}_\omega \tag{4.8c}$$

The first relation of (4.6) immediately follows from (4.8b) and (4.8c).

Remark 4.5. In the proof of the first relation of (4.6), we used the Einstein condition (2.6) only. Therefore, we note that it also holds in X_n.

Theorem 4.6. In ESX_n the vector field U_λ is orthogonal to X_λ (or to S_λ). That is,

$$U_\alpha X^\alpha=U_\alpha S^\alpha=0 \tag{4.9}$$

Proof. In virtue of the second relation of (4.6) and the skew-symmetry of $k_{\lambda\mu}$, the first relation of (4.9) may be proved in the following way:

$$U_\alpha X^\alpha=k_\alpha{}^\beta X_\beta X^\alpha=k_{\alpha\beta}X^\beta X^\alpha=0$$

Therefore, the second relation of (4.9) is obvious in virtue of (4.5).

Theorem 4.7. In ESX_n we have

$$D_\lambda X_\mu = \nabla_\lambda X_\mu - 2U_{(\lambda}X_{\mu)} \tag{4.10a}$$

$$D_{[\lambda}X_{\mu]} = \nabla_{[\lambda}X_{\mu]} = \partial_{[\lambda}X_{\mu]} \tag{4.10b}$$

$$\nabla_{[\lambda}U_{\mu]} = 0, \qquad D_{[\lambda}U_{\mu]} = 2U_{[\lambda}X_{\mu]} \tag{4.10c}$$

Proof. The proof of this theorem follows easily from (2.10), (2.11), and (4.6).

4.2. Parallelism in ESX_n

In virtue of (3.3) and (4.2), a curve C in ESX_n, given by (3.1), is a path if it satisfies

$$\left[\frac{dy^{[\lambda}}{dt}\left(\frac{d^2y^{\nu]}}{dt^2} + \left\{{}^{\nu]}_{\alpha\beta}\right\}\frac{dy^\alpha}{dt}\frac{dy^\beta}{dt} + k_\alpha{}^{\nu]}X_\beta\frac{dy^\alpha}{dt}\frac{dy^\beta}{dt}\right) = 0\right] \tag{4.11}$$

As a consequence of (4.11), we have the following result.

Theorem 4.8. A necessary and sufficient condition for a path C in ESX_n to be a geodesic is that the tangential vector field $T^\alpha = dy^\alpha/dt$ *of* C satisfies the following condition:

$$(X_\alpha T^\alpha)(k_\beta{}^{[\lambda}T^{\nu]}T^\beta) = 0 \tag{4.12}$$

In order to obtain some geometrical consequences of the condition (4.12), we need the following definition.

Definition 4.9. A curve C in ESX_n which is

$$\begin{cases} \text{tangent to a basic vector of } ESX_n \\ \text{orthogonal to the vector } X_\lambda \\ \text{tangent to the vector } U_\lambda \\ \text{tangent to the vector } X_\lambda \end{cases}$$

at each point of C is called

$$\begin{cases} \textit{an } ES_1 \textit{ curve} \\ \textit{an } ES_2 \textit{ curve} \\ \textit{an } ES_3 \textit{ curve} \\ \textit{an } ES_4 \textit{ curve} \end{cases}$$

of ESX_n.

Since the vectors X_λ and U_λ are orthogonal in virtue of (4.9), we note that an ES_3 curve is an ES_2 curve, while an ES_2 curve may not be ES_3.

According to the Definition 4.9, the characterization of an ES_i curve is given by

$$C \text{ is an } ES_1 \text{ curve} \overset{(2.9)}{\Longleftrightarrow} k_\beta{}^{[\lambda}T^{\mu]}T^\beta = 0 \quad \text{along } C \tag{4.13a}$$

$$C \text{ is an } ES_2 \text{ curve} \rightleftarrows X_\alpha T^\alpha = 0 \quad \text{along } C \tag{4.13b}$$

$$C \text{ is an } ES_3 \text{ curve} \rightleftarrows T_\alpha = \rho U_\alpha \quad \text{along } C \quad \text{for a suitable } \rho \neq 0 \tag{4.13c}$$

$$C \text{ is an } ES_4 \text{ curve} \rightleftarrows T_\alpha = \rho X_\alpha \quad \text{along } C \quad \text{for a suitable } \rho \neq 0 \tag{4.13d}$$

Now, we are ready to state and prove consequences of the condition (4.12).

Theorem 4.10. Let C be a path in ESX_n. Then the following statements hold:

(a) If C is a geodesic, it is an ES_1 or ES_2 curve.
(b) If C is an ES_i curve, $i = 1, 2, 3$, then it is at the same time a geodesic.
(c) If C is a minimal ES_4 curve, it is a minimal geodesic.

Proof. Our assertions in this theorem are immediate consequences of (4.13) and Theorem 4.8.

Since there exists a unique ES connection in X_n, it is not possible to consider a change of ES connections which preserves parallelism. However, the following theorem gives the most general parallelism-preserving change of connections one of which is the ES connection.

Theorem 4.11. A necessary and sufficient condition that parallelism be the same along every curve in X_n with respect to two connections one of which is the ES connection $\Gamma_{\lambda\mu}^{\nu}$ is that the other connection $\bar{\Gamma}_{\lambda\mu}^{\nu}$ be given by

$$\bar{\Gamma}_{\lambda\mu}^{\nu} = \{{}_\lambda{}^\nu{}_\mu\} - 2g^\nu{}_{(\lambda}x_{\mu)} + 2\delta_\lambda^\nu A_\mu \tag{4.14}$$

which may be split into

$$\begin{aligned}\bar{\Lambda}_{\lambda\mu}^{\nu} &= \{{}_\lambda{}^\nu{}_\mu\} + 2k_{(\lambda}{}^\nu X_{\mu)} + 2\delta_{(\lambda}{}^\nu p_{\mu)} \\ &= \{{}_\lambda{}^\nu{}_\mu\} - 2g^\nu{}_{(\lambda}X_{\mu)} + 2\delta_{(\lambda}{}^\nu A_{\mu)}\end{aligned} \tag{4.15a}$$

$$\bar{S}_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu A_{\mu]} \tag{4.15b}$$

where

$$A_\lambda = X_\lambda + p_\lambda \tag{4.16}$$

is an arbitrary vector.

Proof. Suppose that parallelism is the same along every curve with respect to two connections one of which is the ES connection $\Gamma_{\lambda\mu}^{\nu}$.

Substituting (2.11) into (3.4) and making use of (4.16) and

$$g^{\nu}{}_{\lambda}=\delta_{\lambda}{}^{\nu}-k_{\lambda}{}^{\nu}$$

we have (4.14). Conversely, suppose that (3.2b) holds for the ES connection $\Gamma^{\nu}_{\lambda\mu}$. Then, in virtue of (3.2b), (4.14), and (4.16), it follows that

$$V^{[\lambda}\left(\frac{dV^{\nu]}}{dt}+\bar{\Gamma}^{\nu]}_{\beta\alpha}V^{\beta}\frac{dy^{\alpha}}{dt}\right)=2V^{[\lambda}\delta_{\beta}{}^{\nu]}V^{\beta}\frac{dy^{\alpha}}{dt}p_{\alpha}=0$$

This proves the converse statement of our assertion.

As an immediate consequence of Theorem 4.11, we have the following theorem simply by putting $A_{\mu}=0$ in (4.14) and (4.15).

Theorem 4.12. Parallelism is preserved along every curve in X_n with respect to the ES connection $\Gamma^{\nu}_{\lambda\mu}$ and a symmetric connection $\bar{\Gamma}^{\nu}_{\lambda\mu}$ given by

$$\bar{\Gamma}^{\nu}_{\lambda\mu}=\{{}_{\lambda}{}^{\nu}{}_{\nu}\}-2g^{\nu}{}_{(\lambda}X_{\mu)} \qquad (4.17)$$

Our final result is the following theorem, which gives a precise tensorial form of the tensor $T_{\omega\mu\nu}$ in ESX_n.

Theorem 4.13. In ESX_n the tensor $T_{\omega\mu\lambda}$ defined by (3.12) is given by

$$T_{\omega\mu\alpha}=(V_{\alpha}-k_{\alpha}{}^{\beta}V_{\beta})X_{[\omega}V_{\mu]}+(V^{\beta}X_{\beta})V_{[\mu}g_{\omega]\alpha} \qquad (4.18)$$

Proof. In virtue of (2.6b), (2.10), and (2.2), we have

$$\begin{aligned} V^{\beta}D_{\alpha}h_{\beta\omega} &= 2V^{\beta}S_{\alpha(\omega}{}^{\gamma}g_{\beta)\gamma} \\ &= 2V^{\beta}(\delta_{\alpha}{}^{\gamma}X_{(\omega}g_{\beta)\gamma}-X_{\alpha}\delta_{(\omega}{}^{\gamma}g_{\beta)\gamma}) \\ &= 2V^{\beta}(X_{(\omega}g_{\beta)\alpha}-X_{\alpha}g_{(\beta\omega)}) \\ &= X_{\omega}V_{\alpha}-(k_{\alpha}{}^{\beta}V_{\beta})X_{\omega}-2X_{\alpha}V_{\omega}+V^{\beta}X_{\beta}g_{\omega\alpha} \qquad (4.19) \end{aligned}$$

Substitution of (4.19) into (3.12) immediately gives (4.18).

5. PARALLELISM IN A SUBMANIFOLD OF ESX_n

In this final section we investigate parallelism, paths, geodesics, and ES_i curves in a submanifold of ESX_n, using the C-nonholonomic frame of reference, and obtain several interesting properties concerning parallelism. In Section 5.1 we discuss parallelism in a general X_n, and then apply the results to a submanifold of ESX_n in Section 5.2. In this section, we particularly investigate the properties of ES_i curves.

5.1. Parallelism in a Submanifold of X_n

All discussions in the present section are restricted to a submanifold X_m of a general X_n.

Let C be a curve in X_m, given by

$$y^\nu = y^\nu(s) \quad \text{in the } y\text{'s} \tag{5.1a}$$

$$x^i = x^i(s) \quad \text{in the } x\text{'s} \tag{5.1b}$$

Let V^ν be the components in the y's of a vector field in X_m defined along C. Then, in virtue of (2.14) and (2.26), its induced components V^i on X_m are related to V^ν as

$$V^\alpha = V^k B^\alpha_k, \qquad V^i = V^\alpha B^i_{\cdot\alpha} \tag{5.2}$$

Along C introduce two vector fields A^ν and C^i by

$$A^\nu = \frac{dy^\alpha}{ds} D_\alpha V^\nu, \qquad C^i = \frac{dx^j}{ds} D_j V^i \tag{5.3}$$

respectively, where D_j denotes the symbol of the covariant derivative with respect to the induced connection Γ^k_{ij} defined by (2.28).

Theorem 5.1. The vector C^i is the induced vector on X_m of the vector A^ν in X_n. That is,

$$C^i = A^i = A^\alpha B^i_{\cdot\alpha} \tag{5.4a}$$

The inverse relation of (5.4a) is given by

$$A^\nu = A^i B^\nu_i + \sum_x A^x \underset{x}{N}{}^\nu \tag{5.4b}$$

Proof. Using the first relation of (5.2) and (2.28), we have

$$\begin{aligned}(D_\beta V^\alpha) B^i_{\cdot\alpha} B^\beta_j &= \left(\frac{\partial V^\alpha}{\partial x^j} + \Gamma^\alpha_{\gamma\beta} V^\gamma B^\beta_j\right) B^i_{\cdot\alpha} \\ &= \left[\frac{\partial V^k}{\partial x^j} B^\alpha_k + V^k (B^\alpha_{kj} + \Gamma^\alpha_{\gamma\beta} B^\gamma_k B^\beta_j)\right] B^i_{\cdot\alpha} \\ &= \frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj} = D_j V^i \end{aligned} \tag{5.5}$$

We now substitute (5.5) into the second relation of (5.3) to prove (5.4a) in the following way

$$C^i = (D_\beta V^\alpha)\left(B_j^\beta \frac{dx^j}{ds}\right)B_\alpha^i$$

$$= (D_\beta V^\alpha)\frac{dy^\beta}{ds} B_\alpha^i = A^\alpha B_\alpha^i = A^i$$

The relation (5.4b) directly follows from (2.32a).

In virtue of (5.4), it should be noted that A^ν is not tangential to X_m in general, whereas V^ν is tangential to X_m.

Definition 5.2. The vector field A^ν (or A^i) is called *the generalized derived vector of* V^ν (or V^i) *in the direction of C with respect to* X_n *or* $\Gamma_{\lambda\mu}^\nu$ (or X_m or Γ_{ij}^k). In particular, when V^ν (or V^i) is the unit tangent vector field to C, we call

$$A^\nu = \frac{dy^\alpha}{ds} D_\alpha \frac{dy^\nu}{ds} \qquad \left(\text{or} \quad A^i = \frac{dx^j}{ds} D_j \frac{dx^i}{ds}\right) \tag{5.6}$$

the generalized first curvature vector of C with respect to X_n (or X_m).

Theorem 5.3. The bivectors $V^{[\lambda}A^{\nu]}$ and $V^{[i}A^{j]}$ are related by

$$V^{[i}A^{j]} = V^{[\alpha}A^{\beta]}B_\alpha^i B_\beta^j \tag{5.7a}$$

$$V^{[\lambda}A^{\nu]} = V^{[i}A^{j]}B_i^\lambda B_j^\nu + \sum_x A^x V^{[\lambda}\underset{x}{N}^{\nu]} \tag{5.7b}$$

Proof. The relation (5.7a) is a direct consequence of (2.33). On the other hand, (2.32a) gives

$$V^{[\lambda}A^{\nu]} = V^{[i}A^{j]}B_i^\lambda B_j^\nu + \sum_x V^{[x}A^{i]}\underset{x}{N}^\lambda B_i^\nu$$

$$+ \sum_x V^{[i}A^{x]}B_i^\lambda \underset{x}{N}^\nu + \sum_{x,y} V^{[x}A^{y]}\underset{x}{N}^\lambda \underset{y}{N}^\nu$$

Since the vector V^ν is tangential to X_m, the last $n-m$ C-nonholonomic components V^x of V^ν vanish. Substituting $V^x=0$ into the above relation and making use of (5.2), we finally have (5.7b).

In the following two theorems, we give geometrical interpretations of the relations (5.7).

Theorem 5.4. Let C be a curve in a submanifold X_m of X_n and let V^ν be a vector field in X_m defined along C. Then the following statements hold:

(a) If V^ν is parallel along C with respect to X_n, then it is also parallel along C with respect to X_m.

(b) If V^ν is parallel along C with respect to X_m, then it is also parallel along C with respect to X_n or its generalized derived vector A^ν in the direction of C is normal to X_m along C.

Proof. In virtue of (3.2a), (5.3), and (5.4a), we first note that $V^{[\lambda}A^{\nu]}=0$ $(V^{[i}A^{j]}=0)$ if and only if V^ν is parallel along C with respect to X_n (X_m). Hence the statement (a) follows directly from (5.7a). Conversely, suppose that V^ν is parallel along C with respect to X_m. Then $V^{[i}A^{j]}=0$ in (5.7b). *If the vector A^ν is tangential to X_m along C*, then $A^x=0$ for all x and so $V^{[\lambda}A^{\nu]}=0$ in virtue of (5.7b). Hence V^ν is parallel along C with respect to X_n in this case. *If the vector A^ν is not tangential to X_m along C*, then $A^x \neq 0$ for some x and hence (5.7b) gives [7]

$$V^{[\lambda}\Big(A^{\nu]}-\sum_x \underset{x}{N}{}^{\nu]}A^x\Big)=0 \rightarrow A^\nu=\sum_x A^x \underset{x}{N}{}^\nu$$

This shows that the vector A^ν is normal to X_m in this case. Hence the statement (b) is proved.

Theorem 5.5. Let C be a curve in a submanifold X_m of X_n. Then the following statements hold:

(a) If C is a path in X_n, then it is also a path in X_m.

(b) If C is a path in X_m, then it is also a path in X_n or its generalized first curvature vector with respect to X_n is normal to X_m along C.

Proof. Taking the vector field V^ν in Theorem 5.4 as the unit tangent vector field to C, that is,

$$V^\nu=\frac{dy^\nu}{ds}, \qquad V^i=\frac{dx^i}{ds} \tag{5.8}$$

our assertion immediately follows from Theorem 5.4.

5.2. ES_i Curves and Parallelism in a Submanifold of ESX_n

Now we are ready to obtain several interesting properties of ES_i curves concerning parallelism in a submanifold X_m of ESX_n, in addition to the results of Section 5.1, *which also hold in an X_m of ESX_n*.

As given in (2.31), the induced connection Γ^k_{ij} of the ES connection is also semisymmetric, but it is not Einstein in general. Therefore, we note that

[7]Another possible case for the left-hand side to hold is that the relation $V^\nu=A^\nu-\sum_x A^x \underset{x}{N}{}^\nu$ holds. In virtue of (5.4b), this relation leads to

$$V^\nu=A^iB_i^\nu \xrightarrow{(5.2)} V^i=A^i$$

which is a contradiction to the Definition 3.1 of parallelism.

a submanifold X_m of ESX_n is not an ES manifold in general. It is an ES manifold if and only if the unified field tensor $g_{\lambda\mu}$ on ESX_n satisfies the condition (2.32).

In order to investigate the properties of ES_i curves on an X_m of ESX_n, we need the following definition similar to the Definition 4.9 or the criterion (4.13):

Definition 5.6. A curve C on an X_m of ESX_n is called

$$\begin{cases} \textit{an } ES_1 \textit{ curve} \\ \textit{an } ES_2 \textit{ curve} \\ \textit{an } ES_3 \textit{ curve} \\ \textit{an } ES_4 \textit{ curve} \end{cases}$$

with respect to X_m, if along C the induced vector T^i of T^ν satisfies the condition

$$\begin{cases} k_i{}^{[j}T^{k]}T^i = 0 \\ X_i T^i = 0 \\ T_i = \rho U_i, \qquad \rho \neq 0 \\ T_i = \rho X_i, \qquad \rho \neq 0 \end{cases}$$

The following theorems give some special properties of ES_i curves on X_m.

Theorem 5.7. Let C be a curve on an X_m of ESX_n. Then the following statements hold:

(a) If C is an ES_i curve with respect to X_n, it is also an ES_i curve with respect to X_m. Here, $i = 1, 2, 3, 4$.

(b) If C is an ES_1 curve with respect to X_m, it is also an ES_1 curve with respect to X_n if the following condition is satisfied:

$$\sum_x k_i{}^x T^h T^i \underset{x}{N}{}^{[\lambda} B_h^{\nu]} = 0 \tag{5.9}$$

(c) If C is an ES_2 curve with respect to X_m, it is also an ES_2 curve with respect to X_n.

(d) If C is an ES_3 curve with respect to X_m, it is also an ES_3 curve with respect to X_n or its tangent T_λ along C is given by

$$T_\lambda = \rho\left(U_\lambda - \sum_x U_x \overset{x}{N}_\lambda\right) \tag{5.10}$$

(e) If C is an ES_4 curve with respect to X_m, it is also an ES_4 curve with respect to X_n or its tangent T_λ along C is given by

$$T_\lambda = \rho\left(X_\lambda - \sum_x X_x \overset{x}{N}_\lambda\right) \tag{5.11}$$

Proof. Since the curve C is on X_m, we first note that

$$T^\alpha \overset{x}{N}_\alpha = 0 \tag{5.12}$$

Making use of (2.19), (2.21), (2.26), and (5.12), we have

$$\begin{aligned} k_i{}^{[j}T^{k]}T^i &= k_\beta{}^\alpha T^\gamma T^\varepsilon B_\alpha^{[j} B_\gamma^{k]}\left(\delta_\varepsilon^\beta - \sum_x \underset{x}{N}{}^\beta \overset{x}{N}_\varepsilon\right) \\ &= k_\beta{}^{[\alpha}T^{\gamma]}T^\beta B_\alpha^j B_\gamma^k \end{aligned} \tag{5.13a}$$

$$\begin{aligned} X_i T^i &= X_\alpha T^\beta B_\beta^\alpha \\ &= X_\alpha T^\beta\left(\delta_\beta^\alpha - \sum_x \underset{x}{N}{}^\alpha \overset{x}{N}_\beta\right) = X_\alpha T^\alpha \end{aligned} \tag{5.13b}$$

$$\begin{aligned} &T_\lambda = \rho U_\lambda \quad \text{for some } \rho \neq 0 \\ &\quad \to (T_i - \rho U_i)B_\lambda^i = 0 \quad \text{for some } \rho \neq 0 \\ &\quad \to T_i = \rho U_i \quad \text{for some } \rho \neq 0 \end{aligned} \tag{5.13c}$$

(since U_λ is tangential to X_m in this case); and

$$\begin{aligned} &T_\lambda = \rho X_\lambda \quad \text{for some } \rho \neq 0 \\ &\quad \to (T_i - \rho X_i)B_\lambda^i = 0 \quad \text{for some } \rho \neq 0 \\ &\quad \to T_i = \rho X_i \quad \text{for some } \rho \neq 0 \end{aligned} \tag{5.13d}$$

(since X_λ is tangential to X_m in this case).

Our statements (a) and (c) immediately follow from (5.13) and Definition 5.6. In order to prove the converse statement (b), we use (2.23) to obtain

$$\begin{aligned} k_\beta{}^\lambda &= k_B{}^A B_A^\lambda B_\beta^B \\ &= k_i{}^j B_j^\lambda B_\beta^i + \sum_x k_i{}^x \underset{x}{N}{}^\lambda B_\beta^i + \sum_x k_x{}^i B_i^\lambda \overset{x}{N}_\beta + \sum_{x,y} k_x{}^y \underset{y}{N}{}^\lambda \overset{x}{N}_\beta \end{aligned} \tag{5.14}$$

Making use of (2.20), (2.26), (5.12), and (5.14), it follows that

$$k_\beta{}^{[\lambda}T^{\nu]}T^\beta = k_i{}^j T^h T^k B_j^{[\lambda} B_h^{\nu]} B^i_\beta B^\beta_k + \sum_x k_i{}^x T^h T^k \overset{x}{N}{}^{[\lambda} B_h^{\nu]} B^i_\beta B^\beta_k$$

$$= k_i{}^{[j}T^{h]} T^i B_j^\lambda B_h^\nu + \sum_x k_i{}^x T^h T^i \overset{x}{N}{}^{[\lambda} B_h^{\nu]} \tag{5.15}$$

Our statement (b) immediately follows from the relation (5.15) in virtue of (4.13a) and Definition 5.6. In order to prove the statement (d), suppose that C is an ES_3 curve with respect to X_m. Then, in virtue of (2.26), we have

$$T_i = \rho U_i \quad \text{for some } \rho \neq 0$$

$$\rightarrow (T_\alpha - \rho U_\alpha) B_i^\alpha = 0 \quad \text{for some } \rho \neq 0$$

The above result implies two cases. The first case is that $T_\alpha = \rho U_\alpha$ for some $\rho \neq 0$. This means that C is an ES_3 curve with respect to X_n. The second case is that the vector $T_\alpha - \rho U_\alpha$ is normal to X_m in virtue of (2.20). That is,

$$T_\alpha - \rho U_\alpha = \sum_x C_x \overset{x}{N}_\alpha \qquad \text{for } C_x \text{ not all zero} \tag{5.16a}$$

But, in virtue of (2.25b), the left-hand side of (5.16a) is

$$T_\alpha - \rho U_\alpha = T_i B^i_\alpha - \rho\left(U_i B^i_\alpha + \sum_x U_x \overset{x}{N}_\alpha \right)$$

$$= (T_i - \rho U_i) B^i_\alpha - \rho \sum_x U_x \overset{x}{N}_\alpha$$

$$= -\rho \sum_x U_x \overset{x}{N}_\alpha \tag{5.16b}$$

which shows that the tangent vector T_λ is given by (5.10) in this case. Hence the statement (d) is proved. The proof of the statement (e) is similar to that of statement (d).

Remark 5.8. As noted in the paragraph following Definition 4.9, if a curve C on X_m is ES_3 with respect to X_n, it is also ES_2 with respect to X_n. Hence, it is both ES_3 and ES_2 with respect to X_m according to the statement (a) of Theorem (5.7). However, a curve C on X_m which is ES_3 with respect to X_m is not ES_2 with respect to X_m in general. Probably, this is due to the fact that X_m is not an ES manifold in general.

Theorem 5.9. Let C be a path of ESX_n which lies on an X_m of ESX_n. Then the following statements hold:

(a) If C is a geodesic with respect to ESX_n, it is an ES_1 or ES_2 curve with respect to both X_m and ESX_n.

(b) If C is an ES_1 curve with respect to X_m and satisfies the condition (5.9), it is a geodesic with respect to ESX_n.

(c) If C is an ES_2 curve with respect to X_m, it is a geodesic with respect to ESX_n.

Proof. First, we note that C *is also a path with respect to* X_m in virtue of Theorem 5.5(a). If a path C is a geodesic with respect to ESX_n, it is an ES_1 or ES_2 curve with respect to ESX_n in virtue of Theorem 4.10(a). Hence, the statement (a) immediately follows from Theorem 5.7(a). The statements (b) and (c) are direct consequences of Theorem 5.7(b), (c) and Theorem 4.10(b).

REFERENCES

Chung, K. T. (1963). *Nuovo Cimento* (X), **27**, 1297–1324.
Chung, K. T., and Kim, M. Y. (1991). *International Journal of Theoretical Physics*, this issue.
Chung, K. T., *et al.* (1968). *Nuovo Cimento* (X), **58B**, 201–212.
Chung, K. T., *et al.* (1969). *Tensor*, **20**, 143–149.
Chung, K. T., *et al.* (1981*a*). *International Journal of Theoretical Physics*, **20**, 739–747.
Chung, K. T., *et al.* (1981*b*). *Journal of Korean Mathematical Society*, **18**, 43–48.
Chung, K. T., *et al.* (1983). *Acta Mathematica Hungarica*, **41**, 47–52.
Chung, K. T., *et al.* (1985). *Acta Mathematica Hungarica*, **45**, 141–149.
Chung, K. T., *et al.* (1987). *Nuovo Cimento*, **100B**, 537–550.
Chung, K. T., *et al.* (1988*a*). *International Journal of Theoretical Physics*, **27**, 1083–1104.
Chung, K. T., *et al.* (1988*b*). *International Journal of Theoretical Physics*, **27**, 1105–1136.
Chung, K. T., *et al.* (1989*a*). *International Journal of Theoretical Physics*, **28**, 851–866.
Chung, K. T., *et al.* (1989*b*). *International Journal of Theoretical Physics*, **28**, 867–873.
Einstein, A. (1950). *The Meaning of Relativity*, Princeton University Press, Princeton, New Jersey.
Hlavatý, V. (1957). *Geometry of Einstein's Unified Field Theory*, Noordhoop Ltd.
Mishra, R. S. (1962). *Tensor*, *N. S.*, **12**, 90.
Wrede, R. C. (1958). *Tensor*, **8**, 95–122.